

# On approximation by Stancu type $q$ -Bernstein-Schurer-Kantorovich operators

M. Mursaleen and Taqseer Khan

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India  
mursaleenm@gmail.com; taqi.khan91@gmail.com;

## Abstract

In this paper we introduce the Stancu type generalization of the  $q$ -Bernstein-Schurer-Kantorovich operators and examine their approximation properties. We investigate the convergence of our operators with the help of the Korovkin's approximation theorem and examine the convergence of these operators in the Lipschitz class of functions. We also investigate the approximation process for these operators through the statistical Korovkin's approximation theorem. Also, we present some direct theorems for these operators. Finally we introduce the bivariate analogue of these operators and study some results for the bivariate case.

*Keywords:* Stancu type  $q$ -Bernstein-Schurer-Kantorovich operators; modulus of continuity; positive linear operators; Korovkin type approximation theorem; statistical approximation, Lipschitz class of functions.

*AMS Subject Classifications (2010):* 40A30, 41A10, 41A25, 41A36,

## 1 Introduction and preliminaries

The  $q$ -calculus has played an important role in the field of approximation theory since last three decades. In the year of 1987, A. Lupas was the first to apply the  $q$ -calculus to approximation theory. He introduced the  $q$ -analogue of the well known Bernstein polynomials [16]. Another remarkable application of the  $q$ -calculus advented in the year of 1997 by Phillips [14]. He used the  $q$ -calculus to define another interesting  $q$ -analogue of the classical Bernstein polynomials. Ostrovska [15] obtained more results on the  $q$ -Bernstein polynomials. In the sequel many researchers have studied the  $q$ -analogues of many well known operators like Baskakov operators, Meyer-König-Zeller operators, Szász-Mirakyan operators, Bleiman, Butzer and Hahn operators (written succinctly as BBH). Also the  $q$ -analogues of some integral operators like Kantorovich and Durrmeyer type were introduced and their approximation properties were studied.

In [20] Muraru defined the  $q$ -Bernstein-Schurer operators in 2011. She used the modulus of continuity to obtain the rate of convergence of the  $q$ -Bernstein-Schurer operators. Recently, the Schurer modifications of some positive linear operators have been studied in [1, 2, 3].

Kantorovich introduced the following integral type generalization of the classical Bernstein operators

$$L_n(f; x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1],$$

is the Bernstein basis function. The approximation properties of these operators through the Korovkin's approximation theorem were studied in [5].

Dalmanoglu presented another Kantorovich type generalization of the  $q$ -Bernstein polynomials and studied some approximation results in [9]. Below we give some rudiments of the  $q$ -calculus. In [10], Radu investigated the statistical convergence properties of the Bernstein-Kantorovich polynomials based on  $q$ -integers. Recently, many researchers have studied various  $q$ -extensions of the Kantorovich operators in [11, 12, 13].

For any fixed real number  $q > 0$  and  $k \in \mathbb{N} \cup \{0\}$ , the  $q$ -integer of  $k$ , denoted by  $[k]_q$ , is defined by

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

and the  $q$ -factorial  $[k]_q!$  is defined as

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \dots [1]_q, & k \geq 1 \\ 1, & k = 0. \end{cases}$$

The  $q$ -concept can be extended to any real number  $k$ . For integers  $n$  and  $k$  such that  $0 \leq k \leq n$ , the  $q$ -analogue of the binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For the  $q$ -binomial coefficient the following relations hold:

$$\begin{aligned} \binom{n}{k}_q &= \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q \\ \binom{n}{k}_q &= q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q. \end{aligned}$$

For  $x \in [0, 1]$  and  $m \in \mathbb{N}^0$ , the  $q$ -analogue of  $(1+x)^n$ , denoted by  $(1+x)_q^n$ , is defined by the polynomial

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\dots(1+q^{n-1}x), & n = 1, 2, 3, \dots \\ 1, & n = 0. \end{cases}$$

For  $0 < q < 1$ ,  $a > 0$ , the  $q$ -definite integral of a real valued function  $f$  is defined by

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \quad a \in \mathbb{R},$$

and over the interval  $[a, b]$ ,  $0 < a < b$ , it is defined by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

For details of these integrals one is referred to [1], [2]. In some situations the above integrals are not appropriate to obtain the  $q$ -analogues of well known integrals. So we use another more general integrals, the Reimann type  $q$ -integrals, defined as follows:

$$\int_a^b f(x) d_q^R x = (1-q)(b-a) \sum_{s=0}^{\infty} f(a + (b-a)q^s) q^s,$$

where  $a, b$  are such that  $0 \leq a < b$  and  $q$  is as above. The later integrals were introduced by Gauchman [3] and Marinković et al. [4].

In this paper, let  $I$  denote the interval  $[0, 1+l]$  equipped with the norm  $\|\cdot\|_{C[0,1+l]}$ , where  $l \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$ .

## 2 Construction of Operators

In 2015, P.N. Agarwal et al. [13] introduced the following Kantorovich type generalization of the  $q$ -Bernstein-Schurer operators

$$K_{n,p}(f; q, x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}^q(x) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q^R t, \quad x \in [0, 1], \quad (2.1)$$

where  $b_{n+p,k}^q(x) = \binom{n+p}{k}_q x^k (1-x)^{n+p-k}$  is the  $q$ -Bernstein basis function. They have investigated the approximation properties of these operators using the Korovkin's approximation theorem. Inspired by their work, we introduce the Stancu type generalisation of the Bernstein-Schurer-Kantorovich operators based on  $q$ -integers as follows:

$$L_{n,l}^{\alpha,\beta}(f; q; x) = ([n+1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} f(t) d_q^R t, \quad x \in [0, 1], \quad (2.2)$$

where  $b_{n,l}^k(q; x) = \binom{n+l}{k}_q x^k (1-x)^{n+l-k}$  is the  $q$ -Bernstein basis function and  $\alpha, \beta$  are such that  $0 < \beta \leq \alpha$ .

$\alpha = 0, \beta = 0$  reduce the operators (2.2) to the operators (2.1). So the newly constructed operators are a generalization of the operators in (2.1). We shall investigate some approximation results for the operators in (2.2). To examine the approximation results, we need the following lemmas.

**Lemma 2.1** *Let  $L_{n,l}^{\alpha,\beta}(f; q; x)$  be given by (2.1). Then the followings hold:*

- (i)  $L_{n,l}^{\alpha,\beta}(1; q; x) = 1,$
- (ii)  $L_{n,l}^{\alpha,\beta}(t; q; x) = \frac{\alpha}{[n+l]_q} + \frac{1}{([n+1]_q + \beta)[2]_q} + \frac{2q[n+l]_q}{([n+1]_q + \beta)[2]_q} x,$
- (iii)  $L_{n,l}^{\alpha,\beta}(t^2; q; x) = \frac{1}{([n+1]_q + \beta)^2 [3]_q} + \frac{2\alpha}{([n+1]_q + \beta)^2 [2]_q} + \frac{\alpha^2}{([n+1]_q + \beta)^2} + \frac{q[n+l]_q((3+4\alpha) + (5+4\alpha)q + 4(1+\alpha)q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} x$   
 $+ \frac{q^2[n+l]_q[n+l-1]_q(1+q+4q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} x^2.$

**Proof.** Before proving the above lemma, we shall first prove the following:

$$\sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^k = 1 - (1-q)[n+l]_q x, \quad (2.3)$$

and

$$\sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{2k} = 1 - (1-q^2)[n+l]_q x + q(1-q)^2[n+l][n+l-1]_q x^2 \quad (2.4)$$

where  $b_{n,l}^k(q; x) = \binom{n+l}{k}_q x^k (1-x)^{n+l-k}.$

In fact we have

$$\begin{aligned}
\sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^k &= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) (1 - 1 + q) [k]_q \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) - (1 - q) [n + l]_q \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q}{[n + l]_q} \\
&= 1 - (1 - q) [n + l]_q \sum_{k=0}^{n+l-1} \binom{n+l-1}{k}_q x^{k+1} (1 - x)_q^{n+l-k-1} \\
&= 1 - (1 - q) [n + l]_q x,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{2k} &= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \left( (1 - 1 + q^2) [k]_q + q(1 - q)^2 [k - 1]_q [k]_q \right) \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) - (1 - q^2) [n + l]_q \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q}{[n + l]_q} + q(1 - q)^2 [n + l]_q [n + l - 1]_q \\
&\quad \times \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q}{[n + l]_q} \frac{[k - 1]_q}{[n - 1]_q} \\
&= 1 - (1 - q^2) [n + l]_q \sum_{k=0}^{n+l} \binom{n+l-1}{k}_q x^k x (1 - x)_q^{n+l-k-1} + q(1 - q)^2 [n + l]_q [n + l - 1]_q \\
&\quad \times \sum_{k=0}^{n+l-2} \binom{n+l-2}{k}_q x^k x^2 (1 - x)_q^{n+l-k-2}.
\end{aligned}$$

(i)

$$\begin{aligned}
\mathbb{L}_{n,l}^{\alpha,\beta}(1; q; x) &= ([n + 1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} 1 d_q^R t \\
&= ([n + 1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} (1 - q) \frac{([k + 1]_q - [k]_q)}{([n + 1]_q + \beta)} \sum_{s=0}^{\infty} q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) = 1.
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathbb{L}_{n,l}^{\alpha,\beta}(t; q; x) &= ([n + 1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} t d_q^R t \\
&= ([n + 1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} (1 - q) \frac{([k + 1]_q - [k]_q)}{([n + 1]_q + \beta)} \sum_{s=0}^{\infty} f \left( \frac{[k]_q + \alpha}{[n + 1]_q + \beta} + \frac{[k + 1]_q - [k]_q}{[n + 1]_q + \beta} q^s \right) q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) (1 - q) \sum_{s=0}^{\infty} \left( \frac{[k]_q + \alpha}{[n + 1]_q + \beta} + \frac{q^k q^s}{[n + 1]_q + \beta} \right) q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \left( \frac{[k]_q + \alpha}{([n + 1]_q + \beta)} + \frac{q^k}{([n + 1]_q + \beta)} \frac{1}{[2]_q} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{[n+l]_q}{[n+1]_q + \beta} \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q + \alpha}{[n+l]_q} + \frac{1}{[2]_q([n+1]_q + \beta)} (1 - (1-q)[n+p]_q x) \\
&= \frac{[n+l]_q}{[n+1]_q + \beta} \left[ \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q}{[n+l]_q} + \sum_{k=0}^{n+l} b_{n,p}^k(q; x) \frac{\alpha}{[n+l]_q} \right] + \frac{1 - (1-q)[n+l]_q x}{[2]_q([n+1]_q + \beta)} \\
&= \frac{[n+l]_q}{[n+1]_q + \beta} \left[ \sum_{k=0}^{n+l} \binom{n+l}{k}_q x^k (1-x)^{n+l-k} \frac{[k]_q}{[n+l]_q} + \frac{\alpha}{[n+l]_q} \right] + \frac{1 - (1-q)[n+l]_q x}{[2]_q([n+1]_q + \beta)} \\
&= \frac{[n+l]_q}{[n+1]_q + \beta} \left[ \sum_{k=1}^{n+l-1} \binom{n+l-1}{k-1}_q x^k (1-x)^{n+l-k-1} + \frac{\alpha}{[n+l]_q} \right] + \frac{1 - (1-q)[n+l]_q x}{[2]_q([n+1]_q + \beta)} \\
&= \frac{[n+l]_q}{[n+1]_q + \beta} x + \frac{\alpha}{[n+l]_q} + \frac{1}{[2]_q([n+1]_q + \beta)} - \frac{(1-q)[n+l]_q x}{[2]_q([n+1]_q + \beta)} \\
&= \left[ \frac{[n+l]_q}{[n+1]_q + \beta} - \frac{(1-q)[n+l]_q}{[2]_q([n+1]_q + \beta)} \right] x + \frac{\alpha}{[n+l]_q} + \frac{1}{[2]_q([n+1]_q + \beta)} \\
&= \frac{\alpha}{[n+l]_q} + \frac{1}{[2]_q([n+1]_q + \beta)} + \frac{2q[n+l]_q}{[2]_q([n+1]_q + \beta)} x.
\end{aligned}$$

(iii)

$$\begin{aligned}
L_{n,l}^{\alpha,\beta}(t^2; q; x) &= ([n+1]_q + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q; x) q^{-k} \frac{[k+1]_q - [k]_q}{([n+1]_q + \beta)} \sum_{s=0}^{\infty} \left( \frac{[k]_q + \alpha}{[n+1]_q + \beta} + \frac{[k+1]_q - [k]_q}{[n+1]_q + \beta} q^s \right)^2 q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) (1-q) \sum_{s=0}^{\infty} \left( \frac{([k]_q + \alpha)^2}{([n+1]_q + \beta)^2} + \frac{q^{2k} q^{2s}}{([n+1]_q + \beta)^2} + \frac{2q^k q^s ([k]_q + \alpha)}{([n+1]_q + \beta)^2} \right) q^s \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \left( \frac{([k]_q + \alpha)^2}{([n+1]_q + \beta)^2} + \frac{2q^k ([k]_q + \alpha)}{([n+1]_q + \beta)^2 (1+q)} + \frac{q^{2k}}{([n+1]_q + \beta)^2 (1+q+q^2)} \right) \\
&= \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{[k]_q^2}{([n+1]_q + \beta)^2} + \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{2\alpha[k]_q}{([n+1]_q + \beta)^2} + \sum_{k=0}^{n+l} b_{n,p}^k(q; x) \frac{\alpha^2}{([n+1]_q + \beta)^2} \\
&\quad + \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{2q^k [k]_q}{([n+1]_q + \beta)^2 [2]_q} + \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{2\alpha q^k}{([n+1]_q + \beta)^2 [2]_q} \\
&\quad + \sum_{k=0}^{n+l} b_{n,l}^k(q; x) \frac{q^{2k}}{([n+1]_q + \beta)^2 [3]_q} \\
&= \frac{[n+l]_q}{[n+1]_q + \beta} x \left( \frac{1}{[n+1]_q + \beta} + q \frac{[n+l-1]_q}{[n+1]_q + \beta} x \right) + \frac{2\alpha[n+l]_q}{([n+1]_q + \beta)^2} x + \frac{\alpha^2}{([n+1]_q + \beta)^2} \\
&\quad + \frac{2q[n+l]_q}{[2]_q([n+1]_q + \beta)^2} x - \frac{2q(1-q)[n+l]_q[n+l-1]_q}{[2]_q([n+1]_q + \beta)^2} x^2 + \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} \\
&\quad \times \left( 1 - (1-q)[n+l]_q x \right) + \frac{1}{[3]_q([n+1]_q + \beta)^2} \\
&\quad \times \left( 1 - (1-q^2)[n+l]_q x + q(1-q)^2[n+l]_q[n+l-1]_q x^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{[3]_q([n+1]_q + \beta)^2} + \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} + \frac{\alpha^2}{([n+1]_q + \beta)^2} + \left[ \frac{[n+l]_q}{([n+1]_q + \beta)^2} + \frac{2\alpha[n+l]_q}{([n+1]_q + \beta)^2} \right. \\
&\quad + \frac{2q[n+l]_q}{[2]_q([n+1]_q + \beta)^2} - \frac{2\alpha(1-q)[n+l]_q}{[2]_q([n+1]_q + \beta)^2} - \frac{(1-q^2)[n+l]_q}{[3]_q([n+1]_q + \beta)^2} \Big] x + \left[ \frac{q[n+l]_q[n+l-1]_q}{([n+1]_q + \beta)^2} \right. \\
&\quad \left. - \frac{2q(1-q)[n+l]_q[n+l-1]_q}{[2]_q([n+1]_q + \beta)^2} + \frac{q(1-q)^2[n+l]_q[n+l-1]_q}{[3]_q([n+1]_q + \beta)^2} \right] x^2 \\
&= \frac{1}{[3]_q([n+1]_q + \beta)^2} + \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} + \frac{\alpha^2}{([n+1]_q + \beta)^2} + \frac{[n+l]_q}{([n+1]_q + \beta)^2 [2]_q [3]_q} \\
&\quad \times \left[ [2]_q [3]_q + 2\alpha [2]_q [3]_q + 2q [3]_q - 2\alpha(1-q) [3]_q - (1-q^2) [2]_q \right] x + \frac{q[n+l]_q[n+l-1]_q}{([n+1]_q + \beta)^2 [2]_q [3]_q} \\
&\quad \times \left[ [2]_q [3]_q - 2(1-q) [3]_q + (1-q)^2 [2]_q \right] x^2 \\
&= \frac{1}{[3]_q([n+1]_q + \beta)^2} + \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} + \frac{\alpha^2}{([n+1]_q + \beta)^2} + \frac{[n+l]_q}{([n+1]_q + \beta)^2 [2]_q [3]_q} \\
&\quad \times q \left( (1+q)(1+2q) + (1+q+q^2)(4\alpha+2) \right) x + \frac{q[n+l]_q[n+l-1]_q}{([n+1]_q + \beta)^2 [2]_q [3]_q} \\
&\quad \times \left( (1+q+q^2)(3q-1) + (1+q^2-2q)(1+q) \right) x^2 \\
&= \frac{1}{[3]_q([n+1]_q + \beta)^2} + \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} + \frac{\alpha^2}{([n+1]_q + \beta)^2} \\
&\quad + \frac{q[n+l]_q((3+4\alpha) + (5+4\alpha)q + 4(1+\alpha)q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} x + \frac{q^2[n+l]_q[n+l-1]_q(1+q+4q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} x^2.
\end{aligned}$$

Hence the lemma.

Remark 2.1. From the Lemma 2.1, we have

$$\begin{aligned}
\text{(i)} \quad L_{n,l}^{\alpha,\beta}((t-x); q; x) &= \left( \frac{2q[n+l]_q}{[2]_q([n+1]_q + \beta)} - 1 \right) x + \frac{1}{[2]_q([n+1]_q + \beta)} + \frac{\alpha}{[n+l]_q}, \\
\text{(ii)} \quad L_{n,l}^{\alpha,\beta}((t-x)^2; q; x) &= \frac{\alpha^2}{([n+1]_q + \beta)^2} + \frac{2\alpha}{([n+1]_q + \beta)^2 [2]_q} + \frac{1}{([n+1]_q + \beta)^2 [3]_q} \\
&\quad + \left( \frac{q[n+l]_q((3+4\alpha) + (5+4\alpha)q + 4(1+\alpha)q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} - \frac{2}{([n+1]_q + \beta)^2 [2]_q} - \frac{2\alpha}{[n+l]_q} \right) x \\
&\quad + \left( \frac{q^2[n+l]_q[n+l-1]_q(1+q+4q^2)}{([n+1]_q + \beta)^2 [2]_q [3]_q} - \frac{4q[n+l]_q}{[n+1]_q + \beta} + 1 \right) x^2.
\end{aligned}$$

**Lemma 2.2** For any  $f \in C(I)$ , we have  $\|L_{n,l}^{\alpha,\beta}(f; q; \cdot)\|_{C[0,1]} \leq \|f\|_{C[0,1+l]}$ .

### 3 Direct Theorems

In this section, we prove some direct theorems for the operators  $L_{n,l}^{\alpha,\beta}(f; q; x)$ .

**Theorem 3.1** Let  $f \in C(I)$  and  $0 < q_n < 1$ . Then the sequence of the operators  $L_{n,l}^{\alpha,\beta}(f; q_n; \cdot)$  converges uniformly to  $f$  on the compact interval  $[0, 1]$  if and only if  $\lim_{n \rightarrow \infty} q_n = 1$ .

**Proof.** (Forward) Suppose that  $\lim_{n \rightarrow \infty} q_n = 1$ . Then we shall show that  $L_{n,l}^{\alpha,\beta}(f; q_n; \cdot)$  converges to  $f$  uniformly on  $[0, 1]$ . Note that for  $0 < q_n < 1$  and  $q_n \rightarrow \infty$  for  $n \rightarrow \infty$ , we get  $[n+1]_{q_n} \rightarrow \infty$

as  $n \rightarrow \infty$ . Now it is easily seen that  $\frac{[n+l]_{q_n}}{[n+1]_{q_n} + \beta} = 1 + q_n^n \frac{([l]_{q_n} - 1)}{[n+1]_{q_n}} - \frac{\beta}{[n+1]_{q_n} + \beta}$ . So when  $n \rightarrow \infty$ ,  $\frac{[n+l]_{q_n}}{[n+1]_{q_n} + \beta} \rightarrow 1$  and  $\frac{[n+l]_{q_n}}{([n+1]_{q_n} + \beta)^2} \rightarrow 0$ . Using this and the Lemma 2.1, we find that  $L_{n,l}^{\alpha,\beta}(1; q_n; x) \rightarrow 1$ ,  $L_{n,l}^{\alpha,\beta}(t; q_n; x) \rightarrow x$  and  $L_{n,l}^{\alpha,\beta}(t^2; q_n; x) \rightarrow x^2$  uniformly on the compact set  $[0, 1]$  as  $n \rightarrow \infty$ . Therefore, the Korovkin's theorem proves that the sequence  $L_{n,l}^{\alpha,\beta}(f; q_n; \cdot)$  converges uniformly to  $f$  on  $[0, 1]$ . We shall prove the converse by the method of contradiction. Suppose that the sequence  $(q_n)$  does not converge to 1. Then there must exist a subsequence  $(q_{n_i})$  of the sequence  $(q_n)$  such that  $q_{n_i} \in (0, 1)$ ,  $q_{n_i} \rightarrow \delta \in [0, 1)$  as  $i \rightarrow \infty$ . Then  $\frac{1}{[n_i+p]_{q_{n_i}}} = \frac{1-q_{n_i}}{1-(q_{n_i})^{n_i+p}} \rightarrow 1-\delta$  as  $i \rightarrow \infty$  because  $(q_{n_i})^{n_i} \rightarrow 0$  as  $i \rightarrow \infty$ . Now if we choose  $n = n_i$ ,  $q = q_{n_i}$  in  $L_{n,l}^{\alpha,\beta}(t; q; x)$  from the Lemma 2.1, then we get  $L_{n,l}^{\alpha,\beta}(t; q; x) = \frac{2\delta}{(1+\delta)(1+\beta(1-\delta))}x + \frac{1-\delta}{1+\delta} \frac{1}{(1+\beta(1-\delta))} + \alpha(1-\delta)$ , which is different from  $x$  when  $i \rightarrow \infty$ , which contradicts our supposition. Hence  $\lim_{n \rightarrow \infty} q_n = 1$ . Hence the theorem is completely proved.

Now we define the following:

Let  $f \in C(I)$ ,  $\delta > 0$  and  $W^2 = \{h : h', h'' \in C(I)\}$ , then the Peetre's K-functional is defined by

$$K_2(f, \delta) = \inf_{h \in W^2} \{\|f - h\| + \delta \|h''\|\},$$

By DeVore and Lorentz theorem [?] there exists a constant  $C > 0$  such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}) \quad (3.1)$$

where  $\omega_2(f, \sqrt{\delta})$ , the second order modulus of continuity of  $f \in C(I)$ , is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < p < \delta^{\frac{1}{2}}} \sup_{x \in I} |f(x+2p) - 2f(x+p) + f(x)|.$$

Also by  $\omega(f, \delta)$ , we denote the first order modulus of continuity of  $f \in C(I)$  defined as

$$\omega(f, \delta) = \sup_{0 < p < \delta} \sup_{x \in I} |f(x+p) - f(x)|$$

Next we prove the following theorem.

**Theorem 3.2** Let  $L_{n,l}^{\alpha,\beta}(f; q; x)$  be the sequence of positive linear operators defined by (2.1) and  $f \in C(I)$ . Let  $(q_n)$  be the sequence with  $0 < q_n < 1$  and  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . Then there exists a constant  $c > 0$  independent of  $n$  and  $x$  such that

$$|L_{n,l}^{\alpha,\beta}(f; q; x) - f(x)| \leq C \omega_2\left(f, \sqrt{\phi_{n,l}^{\alpha,\beta}(q_n; x)}\right) + \omega\left(f, \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)}x - x\right) \quad (3.2)$$

where  $\phi_{n,l}^{\alpha,\beta}(q_n; x) = L_{n,l}^{\alpha,\beta}((t-x)^2; q_n; x) + \left(\frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)}x - x\right)^2$  and  $x \in [0, 1]$ .

**Proof.** Let us define the following operators

$$\bar{L}_{n,l}^{\alpha,\beta}(f; q_n; x) = L_{n,l}^{\alpha,\beta}(f; q_n; x) + f(x) - f\left(\frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)}x\right) \quad (3.3)$$

In the light of the Lemma 2.1, it is easily seen that  $\bar{L}_{n,l}^{\alpha,\beta}(1; q_n; x) = 1$  and  $\bar{L}_{n,l}^{\alpha,\beta}(t; q_n; x) = x$ . Now from the Taylor's formula, for  $g \in W^2$ , we can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du$$

Applying the operators  $\bar{L}_{n,l}^{\alpha,\beta}$  to both sides of the above equation we get

$$\begin{aligned}
\bar{L}_{n,l}^{\alpha,\beta}(g; q_n; x) - g(x) &= g'(x) \bar{L}_{n,l}^{\alpha,\beta}((t-x); q_n; x) + \bar{L}_{n,l}^{\alpha,\beta} \left( \int_x^t (t-u) g''(u) du \right) \\
&= \bar{L}_{n,l}^{\alpha,\beta} \left( \int_x^t (t-u) g''(u) du, q_n; x \right) \\
&= L_{n,l}^{\alpha,\beta} \left( \int_x^t (t-u) g''(u) du, q_n; x \right) \\
&\quad - \int_x^{\frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} x} \left( \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} \right. \\
&\quad \left. + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} x - u \right) g''(u) du
\end{aligned}$$

Therefore, we will have

$$\begin{aligned}
|\bar{L}_{n,l}^{\alpha,\beta}(g; q_n; x) - g(x)| &\leq L_{n,l}^{\alpha,\beta}((t-x)^2; q_n; x) \|g''\|_{C[0,1+l]} + \left( \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} \right. \\
&\quad \left. + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} x - x \right)^2 \|g''\|_{C[0,1+l]} \\
&= \phi_{n,l}^{\alpha,\beta}(q_n; x) \|g''\|_{C[0,1+l]}
\end{aligned}$$

In view of (3.2), we obtain

$$\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq |\bar{L}_{n,l}^{\alpha,\beta}(f - g; q_n; x) - g(x)| + |\bar{L}_{n,l}^{\alpha,\beta}(g; q_n; x) - g(x)| \\
&\quad + \left| f \left( \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{[2]_{q_n}([n+1]_{q_n} + \beta)} + \frac{2q_n[n+l]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} x \right) - f(x) \right|
\end{aligned}$$

Now we have

$$\|\bar{L}_{n,l}^{\alpha,\beta}(f; q_n; x)\| \leq 3\|f\|_{C[0,1+l]},$$

by the Lemma 2.2, so we have

$$\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq 4\|f - g\|_{C[0,1+l]} + \phi_{n,l}^{\alpha,\beta}(q_n; x) \|g''\|_{C[0,1+l]} + \omega \left( f, \left| \frac{1}{([n+1]_{q_n} + \beta)[2]_{q_n}} \right. \right. \\
&\quad \left. \left. + \frac{[n+l]_{q_n}}{([n+1]_{q_n} + \beta)} \frac{2q_n}{[2]_{q_n}} x - x \right| \right).
\end{aligned}$$

On taking the infimum of the right hand side running over all  $g \in W^2$  and using the definition of the Peetre's functional, we get

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq 4K_2(f, \phi_{n,l}^{\alpha,\beta}(q_n; x)) + \omega \left( f, \frac{1}{([n+1]_{q_n} + \beta)[2]_{q_n}} + \frac{[n+l]_{q_n}}{([n+1]_{q_n} + \beta)} \frac{2q_n}{[2]_{q_n}} x - x \right)$$

Now in view of (3.1), we obtain

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq C\omega_2 \left( f, \sqrt{\phi_{n,l}^{\alpha,\beta}(q_n; x)} \right) + \omega \left( f, \left| \frac{1}{([n+1]_{q_n} + \beta)[2]_{q_n}} + \frac{[n+l]_{q_n}}{([n+1]_{q_n} + \beta)} \frac{2q_n}{[2]_{q_n}} x - x \right| \right),$$

and this completes the proof of the theorem.

Now we shall obtain an estimate of the rate of convergence for the operators defined in (2.1) using the Lipschitz-type maximal function defined as follows [?]

For  $x \in [0, 1]$  and  $\xi \in (0, 1]$ , the Lipschitz-type maximal function is defined as

$$\tilde{\omega}_\xi(f, x) = \sup_{t \neq x, t \in [0, 1+l]} \frac{|f(t) - f(x)|}{|t - x|^\xi} \quad (3.4)$$



Now we prove the following theorem

**Theorem 3.3** *Let  $f \in C(I)$ ,  $0 < \xi \leq 1$  and  $q_n \in (0, 1)$  such that  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . Then for every  $x \in [0, 1]$ , we have*

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq \tilde{\omega}_\xi(f, x)(\gamma_{n,l}(q_n; x))^{\frac{\xi}{2}},$$

where  $\gamma_{n,l}(q_n; x) = L_{n,l}^{\alpha,\beta}((t-x)^2; q_n; x)$

**Proof.** In the light of the Lemma (2.1), we have

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq L_{n,l}^{\alpha,\beta}(|f(t) - f(x)|; q_n; x) \leq \tilde{\omega}_\xi(f, x) L_{n,l}^{\alpha,\beta}(|t - x|^\xi; q_n; x) \quad (3.5)$$

and in view of (3.4), we have

$$|f(t) - f(x)| \leq \tilde{\omega}_\xi(f, x)|t - x|^\xi. \quad (3.6)$$

When we use the Hölder's inequality with  $\tilde{p} = \frac{2}{\xi}$  and  $\tilde{q} = \frac{2}{2-\xi}$ , we obtain

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq \tilde{\omega}_\xi(f, x) L_{n,l}^{\alpha,\beta}(|t - x|^2; q_n; x)^{\frac{\xi}{2}} = \tilde{\omega}_\xi(f, x)(\gamma_{n,l}(q_n; x))^{\frac{\xi}{2}},$$

and hence the theorem.

To prove the next theorem we consider the following Lipschitz-type space of functions [?]:

$$\tilde{Lip}_M(s) = \left\{ f \in C(I) : |f(t) - f(x)| \leq M \frac{|t - x|^s}{(t + x)^{\frac{s}{2}}} \right\},$$

where  $M$  is a positive constant and  $0 < s \leq 1$ .

Now we have the following theorem.

**Theorem 3.4** *Let  $f \in \tilde{Lip}_M(s)$ ,  $s \in (0, 1]$  and  $q_n \in (0, 1)$  such that  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . Then for each  $x \in (0, 1]$ , we have*

$$|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| \leq M \left( \frac{\gamma_{n,l}(q_n; x)}{x} \right)^{\frac{s}{2}}$$

where  $\gamma_{n,l}(q_n; x) = L_{n,l}^{\alpha,\beta}((t-x)^2; q_n; x)$ .

**Proof.** Firstly we will prove the result for  $s = 1$ . In fact we have, for  $f \in \tilde{Lip}_M(1)$ ,

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |f(t) - f(x)| d_{q_n}^R t \\ &\leq M([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} \frac{|t - x|}{\sqrt{t + x}} d_{q_n}^R t \end{aligned}$$

Applying the Cauchy-Schwarz inequality and the fact  $\frac{1}{\sqrt{t+x}} \leq \frac{1}{\sqrt{x}}$ , we get

$$\begin{aligned} |L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq \frac{M}{\sqrt{x}} ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |t - x| d_{q_n}^R t \\ &= \frac{M}{\sqrt{x}} L_{n,l}^{\alpha,\beta}(|t - x|; q_n; x) \\ &\leq M \left( \frac{\gamma_{n,l}(q_n; x)}{x} \right)^{\frac{1}{2}}. \end{aligned}$$

Thus the result is established for  $s = 1$ . Next we prove the result for  $0 < s < 1$ . On using the Hölder's inequality twice for  $\tilde{p} = \frac{1}{s}$  and  $\tilde{q} = \frac{1}{1-s}$ , we get

$$\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |f(t) - f(x)| d_{q_n}^R t \\
&\leq \left\{ \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) \left( ([n+1]_{q_n} + \beta) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |f(t) - f(x)| d_{q_n}^R t \right)^{\frac{1}{s}} \right\}^s \\
&\leq \left\{ ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |f(t) - f(x)|^{\frac{1}{s}} d_{q_n}^R t \right\}^s.
\end{aligned}$$

Now as  $f \in \tilde{Lip}_M(s)$ , we obtain

$$\begin{aligned}
|L_{n,l}^{\alpha,\beta}(f; q_n; x) - f(x)| &\leq M \left\{ ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} \frac{|t-x|}{\sqrt{t+x}} d_{q_n}^R t \right\}^s \\
&\leq \frac{M}{x^{\frac{s}{2}}} \left\{ ([n+1]_{q_n} + \beta) \sum_{k=0}^{n+l} b_{n,l}^k(q_n; x) q_n^{-k} \int_{\frac{[k]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}}^{\frac{[k+1]_{q_n} + \alpha}{[n+1]_{q_n} + \beta}} |t-x| d_{q_n}^R t \right\}^s \\
&= \frac{M}{x^{\frac{s}{2}}} (L_{n,l}^{\alpha,\beta}(|t-x|; q_n; x))^s \\
&\leq M \left( \frac{\gamma_{n,l}(q_n; x)}{x} \right)^{\frac{s}{2}}.
\end{aligned}$$

This completes the proof of the theorem.

## 4 Statistical Convergence

In this section we study the  $A$ -statistical convergence of the operators defined in (2.1) through the Korovkin-type statistical approximation theorem.

Let  $A = (a_{nk})$  be a non-negative infinite summability matrix. Then for a sequence  $x = (x_k)$ , we define the  $A$ -transform of  $x$  as  $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$  whenever the series converges for each  $n$ . We denote it by  $Ax = ((Ax)_n)$ .  $A$  is said to be regular if  $\lim_n (Ax)_n = L$ , whenever  $\lim_n x_n = L$ . The sequence  $x = (x_n)$  is said to be  $A$ -statistically convergent to the limit  $L$ , denoted by  $st_A - \lim_n x_n = L$ , if for each  $\epsilon > 0$ ,  $\lim_n \sum_{k: |x_k - L| \geq \epsilon} a_{nk} = 0$ . If we take the matrix  $A$  to be the Cesàro matrix  $C$  of order one, then the notion of the  $A$ -statistically convergence is same as the statistical convergence.

Now to prove a theorem, we take a sequence  $(q)_n$  such that  $q_n \in (0, 1)$  satisfying the following:  $st_A - \lim_n x_n = 1$ ,  $st_A - \lim_n (q_n)^n = a \in (0, 1)$ ,  $st_A - \lim_n \frac{1}{[n]_{q_n}} = 0$ .

**Theorem 4.1** *Let  $A = (a_{nk})$  be a non-negative regular summability matrix and  $(q_n)$  be a sequence satisfying the above conditions. Then for any  $f \in C(I)$ , we have*

$$st_A - \lim_n \|L_{n,l}^{\alpha,\beta}(f; q; \cdot) - f\|_{C[0,1]} = 0.$$

**Proof.** Let  $e_i(x) = x^i$ , where  $x \in [0, 1]$ ,  $i = 0, 1, 2$ . Then from the Lemma 2.1, we have

$$st_A - \lim_n \|L_{n,l}^{\alpha,\beta}(e_0; q; \cdot) - e_0\|_{C[0,1]} = 0. \quad (4.1)$$

Next, again from the Lemma 2.1, we have

$$\lim_n \|L_{n,l}^{\alpha,\beta}(e_1; q_n; \cdot) - e_1\|_{C[0,1]} \leq \left| \frac{\alpha}{[n+l]_q} + \frac{1}{([n+1]_q + \beta)[2]_q} \right| + \left| \frac{2q[n+l]_q}{([n+1]_q + \beta)[2]_q} - 1 \right|.$$

Now since  $st_A - \lim_n q_n = 1$ ,  $st_A - \lim_n (q_n)^n = a \in (0, 1)$  and  $st_A - \lim_n \frac{1}{[n]_{q_n}} = 0$ , we have

$$st_A - \lim_n \left( \frac{\alpha}{[n+l]_q} + \frac{1}{([n+1]_q + \beta)[2]_q} \right) = 0$$

and

$$st_A - \lim_n \left( \frac{2q_n}{[2]_{q_n}} \frac{1 - q_n^{n+l}}{1 - q_n^{n+1} + \beta(1 - q_n)} - 1 \right) = 0$$

Now, for a given  $\epsilon > 0$ , let us define the following sets:

$$U = \left\{ n \in N : \|L_{n,l}^{\alpha,\beta}(e_1; q_n; \cdot) - e_1\|_{C[0,1]} \geq \epsilon \right\},$$

$$U_1 = \left\{ n \in N : \frac{\alpha}{[n+l]_{q_n}} + \frac{1}{([n+1]_{q_n} + \beta)[2]_{q_n}} \geq \frac{\epsilon}{2} \right\},$$

and

$$U_2 = \left\{ n \in N : \frac{2q_n}{[2]_{q_n}} \frac{1 - q_n^{n+l}}{1 - q_n^{n+1} + \beta(1 - q_n)} - 1 \geq \frac{\epsilon}{2} \right\}.$$

The containment  $U \subseteq U_1 \cup U_2$  is obvious which in turn implies that  $\sum_{n \in U} a_{nk} \leq \sum_{n \in U_1} a_{nk} + \sum_{n \in U_2} a_{nk}$ , and hence we have

$$st_A - \lim_n \|L_{n,l}^{\alpha,\beta}(e_1; q_n; \cdot) - e_1\|_{C[0,1]}. \quad (4.2)$$

Further, using the Lemma 2.1, we have

$$\begin{aligned} \|L_{n,l}^{\alpha,\beta}(e_2; q_n; \cdot) - e_2\|_{C[0,1]} \leq & \left| \frac{1}{([n+1]_{q_n} + \beta)^2[3]_{q_n}} + \frac{2\alpha}{([n+1]_q + \beta)^2[2]_{q_n}} + \frac{\alpha^2}{([n+1]_{q_n} + \beta)^2} \right| \\ & + \left| \frac{q_n}{[2]_{q_n}[3]_{q_n}} \frac{((3+4\alpha) + (5+4\alpha)q_n + 4(1+\alpha)q_n^2)(1 - q_n^{n+l})(1 - q_n)}{(1 - q_n^{n+1} + \beta(1 - q_n))^2} \right| \\ & + \left| \frac{q_n^2(1 + q_n + 4q_n^2)}{[2]_{q_n}[3]_{q_n}} \frac{1 - q_n^{n+l}}{(1 - q_n^{n+1} + \beta(1 - q_n))} \frac{1 - q_n^{n+l-1}}{(1 - q_n^{n+1} + \beta(1 - q_n))} - 1 \right|. \end{aligned}$$

Keeping in view  $st_A - \lim_n q_n = 1$ ,  $st_A - \lim_n (q_n)^n = a \in (0, 1)$  and  $st_A - \lim_n \frac{1}{[n]_{q_n}} = 0$ , we obtain

$$\begin{aligned} st_A - \lim_n \left( \frac{1}{([n+1]_{q_n} + \beta)^2[3]_{q_n}} + \frac{2\alpha}{([n+1]_q + \beta)^2[2]_{q_n}} + \frac{\alpha^2}{([n+1]_{q_n} + \beta)^2} \right) &= 0, \\ st_A - \lim_n \left( \frac{q_n}{[2]_{q_n}[3]_{q_n}} \frac{((3+4\alpha) + (5+4\alpha)q_n + 4(1+\alpha)q_n^2)(1 - q_n^{n+l})(1 - q_n)}{(1 - q_n^{n+1} + \beta(1 - q_n))^2} \right) &= 0, \end{aligned}$$

and

$$st_A - \lim_n \left( \frac{q_n^2(1 + q_n + 4q_n^2)}{[2]_{q_n}[3]_{q_n}} \frac{1 - q_n^{n+l}}{(1 - q_n^{n+1} + \beta(1 - q_n))} \frac{1 - q_n^{n+l-1}}{(1 - q_n^{n+1} + \beta(1 - q_n))} - 1 \right) = 0.$$

Now for each  $\epsilon > 0$ , we define the following sets:

$$V = \left\{ n \in N : \|L_{n,l}^{\alpha,\beta}(e_2; q_n; \cdot) - e_2\|_{C[0,1]} \geq \epsilon \right\},$$

$$V_1 = \left\{ n \in N : \frac{1}{([n+1]_{q_n} + \beta)^2[3]_{q_n}} + \frac{2\alpha}{([n+1]_{q_n} + \beta)^2[2]_{q_n}} + \frac{\alpha^2}{([n+1]_{q_n} + \beta)^2} \geq \frac{\epsilon}{3} \right\},$$

$$V_2 = \left\{ n \in N : \frac{q_n}{[2]_{q_n}[3]_{q_n}} \frac{((3+4\alpha) + (5+4\alpha)q_n + 4(1+\alpha)q_n^2)(1-q_n^{n+l})(1-q_n)}{(1-q_n^{n+1} + \beta(1-q_n))^2} \geq \frac{\epsilon}{3} \right\},$$

and

$$V_3 = \left\{ n \in N : \frac{q_n^2(1+q_n+4q_n^2)}{[2]_{q_n}[3]_{q_n}} \frac{1-q_n^{n+l}}{(1-q_n^{n+1} + \beta(1-q_n))} \frac{1-q_n^{n+l-1}}{(1-q_n^{n+1} + \beta(1-q_n))} - 1 \geq \frac{\epsilon}{3} \right\},$$

It is obvious that  $V \subseteq V_1 \cup V_2 \cup V_3$ , which in turn implies that  $\sum_{n \in V} a_{nk} \leq \sum_{n \in V_1} a_{nk} + \sum_{n \in V_2} a_{nk} + \sum_{n \in V_3} a_{nk}$ . Therefore, we get

$$st_A - \lim_n \|L_{n,l}^{\alpha,\beta}(e_2; q_n; \cdot) - e_2\|_{C[0,1]} = 0. \quad (4.3)$$

Now on combining (4.1)-(4.3), the theorem follows from the Korovkin-type statistical approximation theorem as proved in [?]. Hence the proof is complete.

## 5 Construction of the Bivariate Operators

In what follows we construct the bivariate extension of the operators defined by (2.1).

Let  $I_1 = [0, 1+l_1]$  and  $I_2 = [0, 1+l_2]$ . We consider  $C(I_1 \times I_2)$ , the space of all real valued continuous functions defined on  $I_1 \times I_2$  equipped with the following norm

$$\|f\|_{C(I_1 \times I_2)} = \sup_{(x,y) \in I_1 \times I_2} |f(x,y)|.$$

We define the bivariate generaliation of the operators in (2.1) as follows

$$\begin{aligned} L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f(t, s); q_1, q_2; x, y) &= ([n_1+1]_{q_1} + \beta_1)([n_2+1]_{q_2} + \beta_2) \sum_{k_1=0}^{n_1+l_1} \sum_{k_2=0}^{n_2+l_2} q_1^{-k_1} q_2^{-k_2} \\ &\quad \times b_{n_1, n_2; l_1, l_2}^{k_1, k_2}(q_1, q_2; x, y) \int_{\frac{[k_1]_{q_1} + \alpha_1}{[n_1+1]_{q_1} + \beta_1}}^{\frac{[k_1+1]_{q_1} + \alpha_1}{[n_1+1]_{q_1} + \beta_1}} \int_{\frac{[k_2]_{q_2} + \alpha_2}{[n_2+1]_{q_2} + \beta_2}}^{\frac{[k_2+1]_{q_2} + \alpha_2}{[n_2+1]_{q_2} + \beta_2}} f(t, s) d_{q_1}^R t d_{q_2}^R s, \end{aligned}$$

where

$$b_{n_1, n_2; l_1, l_2}^{k_1, k_2}(q_1, q_2; x, y) = \binom{n_1+l_1}{k_1}_{q_1} \binom{n_2+l_2}{k_2}_{q_2} x^{k_1} y^{k_2} (1-x)_{q_1}^{n_1+l_1-k_1} (1-y)_{q_2}^{n_2+l_2-k_2},$$

$f \in C(I_1 \times I_2)$ ,  $0 < q_1, q_2 < 1$ ,  $(x, y) \in [0, 1] \times [0, 1] = J^2$  and  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are such that  $0 < \beta_1 \leq \alpha_1; 0 < \beta_2 \leq \alpha_2$ .

Now we prove a lemma concerning the above operators.

**Lemma 5.1** *Let  $(t, s) \in (I_1 \times I_2)$ ,  $(i, j) \in N^0 \times N^0$  with  $i+j \leq 2$ , and  $t^i s^j$  by  $e_{ij}(t, s)$  be the two dimensional test functions. Then the following equalities hold for the bivariate operators of (?):*

- (i)  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(e_{00}; q_1, q_2; x, y) = 1,$
- (ii)  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(e_{10}; q_1, q_2; x, y) = \frac{\alpha_1}{[n_1+1]_{q_1}} + \frac{1}{([n_1+1]_{q_1} + \beta_1)[2]_{q_1}} + \frac{2q_1[n_1+l_1]_{q_1}}{([n_1+1]_{q_1} + \beta_1)[2]_{q_1}} x,$
- (iii)  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(e_{01}; q_1, q_2; x, y) = \frac{\alpha_2}{[n_2+1]_{q_2}} + \frac{1}{([n_2+1]_{q_2} + \beta_2)[2]_{q_2}} + \frac{2q_2[n_2+l_2]_{q_2}}{([n_2+1]_{q_2} + \beta_2)[2]_{q_2}} y,$
- (iv)  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(e_{20}; q_1, q_2; x, y) = \frac{1}{([n_1+1]_{q_1} + \beta_1)^2 [3]_{q_1}} + \frac{2\alpha_1}{([n_1+1]_{q_1} + \beta_1)^2 [2]_{q_1}} + \frac{\alpha_1^2}{([n_1+1]_{q_1} + \beta_1)^2} \\ + \frac{q_1[n_1+l_1]_{q_1}((3+4\alpha_1) + (5+4\alpha_1)q_1 + 4(1+\alpha_1)q_1^2)}{([n_1+1]_{q_1} + \beta_1)^2 [2]_{q_1} [3]_{q_1}} x + \frac{q_1^2[n_1+l_1]_{q_1}[n_1+l_1-1]_{q_1}(1+q_1+4q_1^2)}{([n_1+1]_{q_1} + \beta_1)^2 [2]_{q_1} [3]_{q_1}} x^2,$

$$(v) \quad L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(e_{02}; q_1, q_2; x, y) = \frac{1}{([n_2+1]_{q_2} + \beta_2)^2 [3]_{q_2}} + \frac{2\alpha_2}{([n_2+1]_{q_2} + \beta_2)^2 [2]_{q_2}} + \frac{\alpha_2^2}{([n_2+1]_{q_2} + \beta_2)^2} \\ + \frac{q_2 [n_2 + l_2]_{q_2} ((3+4\alpha_2) + (5+4\alpha_2)q_2 + 4(1+\alpha_2)q_2^2)}{([n_2+1]_{q_2} + \beta_1)^2 [2]_{q_2} [3]_{q_2}} y + \frac{q_2^2 [n_2 + l_2]_{q_2} [n_2 + l_2 - 1]_{q_2} (1+q_2+4q_2^2)}{([n_2+1]_{q_2} + \beta_2)^2 [2]_{q_2} [3]_{q_2}} y^2.$$

**Proof.** In the light of the Lemma 2.1 and noting that

$$L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(t^i s^j; q_1, q_2; x, y) = L_{n_1, l_1}^{\alpha_1, \beta_1}(t^i; q_1, x) \times L_{n_2, l_2}^{\alpha_2, \beta_2}(s^j; q_2, y), \quad \text{for } 0 \leq i, j \leq 2.$$

the proof is plain and straightforward. So we omit the details.

Now To establish the next theorem we first define the following:

Let  $f \in C(I_1 \times I_2)$  and  $\delta_1, \delta_2 > 0$ . Then the first order complete modulus of continuity for the bivariate case, denoted by  $\omega(f; \delta_1, \delta_2)$  is defined as follows:

$$\omega(f, \delta_1, \delta_2) = \sup\{|f(t, s) - f(x, y)| : |t - x| \leq \delta_1, |s - y| \leq \delta_2\}.$$

Two chief properties of  $\omega(f, \delta_1, \delta_2)$  are as follows:

- (i)  $\omega(f, \delta_1, \delta_2) \rightarrow 0$  as  $\delta_1 \rightarrow 0$  and  $\delta_2 \rightarrow 0$ , and
- (ii)  $|f(t, s) - f(x, y)| \leq \omega(f, \delta_1, \delta_2) \left(1 + \frac{|t-x|}{\delta_1}\right) \left(1 + \frac{|s-y|}{\delta_2}\right).$

Next we state and prove a theorem regarding the rate of convergence of the bivariate operators. To do it we consider a sequence  $(q_{n_i})$  with  $q_{n_i} \in (0, 1)$  such that  $q_{n_i} \rightarrow 1$  and  $q_{n_i}^{n_i} \rightarrow a_i$ ,  $(0 \leq a_i < 1)$  as  $n_i \rightarrow \infty$  for  $i = 1, 2$ . Also we denote  $L_{n_1, l_1}^{\alpha_1, \beta_1}(t - x)^2; q_{n_1}, x)$  and  $L_{n_2, l_2}^{\alpha_2, \beta_2}(s - y)^2; q_{n_2}, y)$  by  $\delta_{n_1}(x)$  and  $\delta_{n_2}(y)$  respectively. Now we have the following theorem:

**Theorem 5.2** *Let  $f \in C(I_1 \times I_2)$ . Then for all  $(x, y) \in J^2$ , we have*

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq 4\omega(f, (\delta_{n_1}(x))^{\frac{1}{2}}, (\delta_{n_2}(y))^{\frac{1}{2}}).$$

**Proof.** Using the fact that the operators  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y)$  are linear and positive together with the property (ii) of the modulus of continuity, we obtain

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|f(t, s) - f(x, y)|; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \\ \leq \omega(f; (\delta_{n_1}(x))^{\frac{1}{2}}, (\delta_{n_2}(y))^{\frac{1}{2}}) \left( L_{n_1, l_1}^{\alpha_1, \beta_1}(1; q_{n_1}, x) + \frac{1}{(\delta_{n_1}(x))^{\frac{1}{2}}} L_{n_1, l_1}^{\alpha_1, \beta_1}(|t - x|; q_{n_1}, x) \right) \\ \times \left( L_{n_2, l_2}^{\alpha_2, \beta_2}(1; q_{n_2}, y) + \frac{1}{(\delta_{n_2}(y))^{\frac{1}{2}}} L_{n_2, l_2}^{\alpha_2, \beta_2}(|s - y|; q_{n_2}, y) \right).$$

On applying the Cauchy-Schwartz inequality,

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq \omega(f; (\delta_{n_1}(x))^{\frac{1}{2}}, (\delta_{n_2}(y))^{\frac{1}{2}}) \\ \times \left( 1 + \frac{1}{(\delta_{n_1}(x))^{\frac{1}{2}}} (L_{n_1, l_1}^{\alpha_1, \beta_1}((t - x)^2; q_{n_1}, x))^{\frac{1}{2}} \right) \\ \times \left( 1 + \frac{1}{(\delta_{n_2}(y))^{\frac{1}{2}}} (L_{n_2, l_2}^{\alpha_2, \beta_2}((s - y)^2; q_{n_2}, y))^{\frac{1}{2}} \right),$$

we obtain the required result.

## 6 Approximation results

In this section we prove some theorems regarding the degree of approximation for the bivariate operators through the Lipschitz class. The Lipschitz class for the bivariate case, denoted by  $Lip_M(\alpha_1, \alpha_2)$

is defined as under:

Let  $0 < \alpha_1, \alpha_2 \leq 1$ . A function  $f$  is said to be in the class  $Lip_M(\alpha_1, \alpha_2)$  if it satisfies the following inequality:

$$|f(x, y) - f(x', y')| \leq M|x - x'|^{\alpha_1}|y - y'|^{\alpha_2}$$

for all  $(x, y), (x', y') \in I_1 \times I_2$ . Now we have the following theorem:

**Theorem 6.1** *Let  $f \in Lip_M(\alpha_1, \alpha_2)$ . Then*

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq M\sqrt{(\delta_{n_1}(y))^{\alpha_1}}\sqrt{(\delta_{n_2}(y))^{\alpha_2}}$$

holds for all  $(x, y) \in J^2$ .

**Proof.** Using the hypothesis, we can write

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|f(t, s) - f(x, y)|; q_{n_1}, q_{n_2}; x, y) \\ &\leq ML_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|t - x|^{\alpha_1}|s - y|^{\alpha_2}; q_{n_1}, q_{n_2}; x, y) \\ &= ML_{n_1; l_1}^{\alpha_1; \beta_1}(|t - x|^{\alpha_1}; q_{n_1}; x)L_{n_2; l_2}^{\alpha_2; \beta_2}(|s - y|^{\alpha_2}; q_{n_2}; y). \end{aligned}$$

Now we apply the Hölder's inequality with  $u_1 = \frac{2}{\alpha_1}, v_1 = \frac{2}{2-\alpha_1}$  and  $u_2 = \frac{2}{\alpha_2}, v_2 = \frac{2}{2-\alpha_2}$  to get

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq ML_{n_1; l_1}^{\alpha_1; \beta_1}((t - x)^2; q_{n_1}; x)^{\frac{\alpha_1}{2}} L_{n_1; l_1}^{\alpha_1; \beta_1}(1; q_{n_1}; x)^{\frac{2-\alpha_1}{2}} \\ &\quad L_{n_2; l_2}^{\alpha_2; \beta_2}((s - y)^2; q_{n_2}; y)^{\frac{\alpha_2}{2}} L_{n_2; l_2}^{\alpha_2; \beta_2}(1; q_{n_2}; y)^{\frac{2-\alpha_2}{2}} \\ &= M(\delta_{n_1}(x))^{\frac{\alpha_1}{2}}(\delta_{n_2}(y))^{\frac{\alpha_2}{2}} \\ &= M\sqrt{(\delta_{n_1}(y))^{\alpha_1}}\sqrt{(\delta_{n_2}(y))^{\alpha_2}} \end{aligned}$$

and this completes the proof of the theorem.

In the ensuing, we will use the following notations:

$$C^1(I_1 \times I_2) = \{f \in C(I_1 \times I_2) : f'_x, f'_y \in C(I_1 \times I_2)\}.$$

Now we will prove the following theorem:

**Theorem 6.2** *Let  $f \in C^1(I_1 \times I_2)$  and  $(x, y) \in J^2$ . Then*

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq \|f'_x\|_{C(I_1 \times I_2)}\sqrt{\delta_{n_1}(x)} + \|f'_y\|_{C(I_1 \times I_2)}\sqrt{\delta_{n_2}(y)}.$$

**Proof.** For a fixed  $(x, y) \in J^2$ , let us write

$$f(t, s) - f(x, y) = \int_x^t f'_u(u, s)d_q u + \int_y^s f'_v(x, v)d_q v.$$

Operating  $L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}$  on both sides of the above equation, we get

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}\left(\left|\int_t^x f'_u(u, s)d_q u\right|; q_{n_1}, q_{n_2}; x, y\right) \\ &\quad + L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}\left(\left|\int_y^s f'_v(x, v)d_q v\right|; q_{n_1}, q_{n_2}; x, y\right). \end{aligned}$$

Now since

$$\left|\int_t^x f'_u(u, s)d_q u\right| \leq \|f'_x\|_{C(I_1 \times I_2)}|t - x|,$$

and

$$|\int_y^s |f'_v(x, v)| d_q v| \leq \|f'_y\|_{C(I_1 \times I_2)} |s - y|,$$

we obtain

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq \|f'_x\|_{C(I_1 \times I_2)} L_{n_1, l_1}^{\alpha_1, \beta_1}(|t - x|; q_{n_1}; x) + \|f'_y\|_{C(I_1 \times I_2)} L_{n_2, l_2}^{\alpha_2, \beta_2}(|s - y|; q_{n_2}; y).$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq \|f'_x\|_{C(I_1 \times I_2)} \left( L_{n_1, l_1}^{\alpha_1, \beta_1}((t - x)^2; q_{n_1}; x) \right)^{\frac{1}{2}} \left( L_{n_1, l_1}^{\alpha_1, \beta_1}(1; q_{n_1}; x) \right)^{\frac{1}{2}} \\ &+ \|f'_y\|_{C(I_1 \times I_2)} \left( L_{n_2, l_2}^{\alpha_2, \beta_2}((s - y)^2; q_{n_2}; y) \right)^{\frac{1}{2}} \left( L_{n_2, l_2}^{\alpha_2, \beta_2}(1; q_{n_2}; y) \right)^{\frac{1}{2}} \\ &= \|f'_x\|_{C(I_1 \times I_2)} \sqrt{\delta_{n_1}(x)} + \|f'_y\|_{C(I_1 \times I_2)} \sqrt{\delta_{n_2}(y)}. \end{aligned}$$

Hence the theorem.

Now we define the following:

If  $f \in C(I_1 \times I_2)$ ,  $\delta > 0$ , then the partial moduli of continuity of  $f$  with respect to  $s$  and  $t$ , is defined by

$$\tilde{\omega}_1(f; \delta) = \sup\{|f(x_1, t) - f(x_2, t)| : t \in I_2 \text{ and } |x_1 - x_2| \leq \delta\}$$

and

$$\tilde{\omega}_2(f; \delta) = \sup\{|f(s, y_1) - f(s, y_2)| : s \in I_1 \text{ and } |y_1 - y_2| \leq \delta\}.$$

Overtly they satisfy the properties of the usual modulus of continuity.

Next we have the following theorem:

**Theorem 6.3** *Let  $f \in C(I_1 \times I_2)$  and  $(x, y) \in J^2$ . Then*

$$|L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| \leq 2\tilde{\omega}_1(f; \sqrt{\delta_{n_1}(x)}) + 2\tilde{\omega}_2(f; \sqrt{\delta_{n_2}(y)})$$

*holds.*

**Proof.** Making use of the definition of partial moduli of continuity, we obtain

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|f(t, s) - f(x, y)|; q_{n_1}, q_{n_2}; x, y) \\ &\leq L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|f(t, s) - f(t, y)|; q_{n_1}, q_{n_2}; x, y) \\ &+ L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(|f(t, y) - f(x, y)|; q_{n_1}, q_{n_2}; x, y) \\ &\leq \tilde{\omega}_1(f; \delta_{n_1}(x)) (L_{n_1, l_1}^{\alpha_1, \beta_1}(1; q_{n_1}; x) + \frac{1}{\sqrt{\delta_{n_1}(x)}} L_{n_1, l_1}^{\alpha_1, \beta_1}(|t - x|; q_{n_1}; x)) \\ &+ \tilde{\omega}_2(f; \delta_{n_2}(y)) (L_{n_2, l_2}^{\alpha_2, \beta_2}(1; q_{n_2}; y) + \frac{1}{\sqrt{\delta_{n_2}(y)}} L_{n_2, l_2}^{\alpha_2, \beta_2}(|s - y|; q_{n_2}; y)) \end{aligned}$$

On using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} |L_{n_1, n_2; l_1, l_2}^{\alpha_1, \alpha_2; \beta_1, \beta_2}(f; q_{n_1}, q_{n_2}; x, y) - f(x, y)| &\leq \tilde{\omega}_1(f; \delta_{n_1}(x)) \left(1 + \frac{1}{\sqrt{\delta_{n_1}(x)}} L_{n_1, l_1}^{\alpha_1, \beta_1}((t - x)^2; q_{n_1}; x)\right) \\ &+ \tilde{\omega}_2(f; \delta_{n_2}(y)) \left(1 + \frac{1}{\sqrt{\delta_{n_2}(y)}} L_{n_2, l_2}^{\alpha_2, \beta_2}((s - y)^2; q_{n_2}; y)\right) \\ &= 2\tilde{\omega}_1(f; \sqrt{\delta_{n_1}(x)}) + 2\tilde{\omega}_2(f; \sqrt{\delta_{n_2}(y)}), \end{aligned}$$

and the proof is completed.

# References

- [1] P.N. Agarwal, V. Gupta, A. Satish Kumar, On  $q$ -analogue of Bernstein-Schurer-Stancu operators, Appl. Math. Comput. 219(14)(2013) 7754-7764.
- [2] P.N. Agarwal, A. Satish Kumar, T.A.K. Sinha, Stancu type generalization of modified Schurer operators based on  $q$ -integers, Appl. Math. Comput. 226(2014)765-776.
- [3] M. Ren, X.M. Zeng, King type modification of  $q$ -Bernstein-Schurer operators, Czechoslovak Math. J. 63(138)(2013) 805-817.
- [4] J. Thomae, Beitrage zur Theorie der durch die Heinsche Reihe. J. Reine. Angew. Math. 70, 258-281 (1869).
- [5] F.H. Jackson, On a  $q$ -definite integrals. Q. J. Pure Appl. Math. 41, 193-203 (1910).
- [6] H. Gauchman, Integral inequalities in  $q$ -calculus, Comput. Math. Appl. 47 (2004) 281-300.
- [7] S. Marinković, P. Rajković, M. Stanković, The inequalities for some types of  $q$ -integrals, Comput. Math. Appl. 56(2008) 2490-2498.
- [8] F. Altomare, M. Campiti, *Korovkin type approximation theory and its applications*, de Gruyter Studies in Mathematics, vol. 17, Walter de Gruyter, Berlin, New York, 1994.
- [9] A. Aral, *Generalization of Szász- Mirakjan operators based on  $q$ -integers*, Math. Comput. Modelling 49, no. 9, (2008), 1052-1062.
- [10] Ç. Atakut, İ. Büyükyazıcı, *Stancu type generalisation of the Favard-Szász operators*, Appl. Math. Lett., 23(2010), 1479-1482.
- [11] Ç. Atakut, N. İspir, *The order approximation by certain linear positive operators*, Math. Balkanica 15(1-2)(2001) 25-33.
- [12] Ö. Dalmanoglu, Approximation by Kantorovich type  $q$ -Bernstein operators, in: Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, 2007, pp. 113-117, ISSN: 1790-2769.
- [13] P.N. Agarwal, Zoltán Finta, A Satish Kumar, Bernstein-Schurer-Kantorovich operators based on  $q$ -integers, Applied Mathematics and Computations 256(2015) 222-231.
- [14] G.M. Phillips, Bernstein polynomials based on the  $q$ -integers, Ann. Numer. Math. 4(1997)511-518.
- [15] S. Ostrovska,  $q$ -Bernstein polynomials and their iterates, J. Approx. Theory 123(2(2003)232-255.
- [16] A. Lupaş, A  $q$ -analogue of the Bernstein operator, Rocky Mountain J. Math. 36(5)(2006)1615-1629.
- [17] C. Radu, Statistical approximation properties of Kantorovich operators based on  $q$ -integers, Creat. Math. Inform. 17 (2)(2008) 75-84.
- [18] Ö. Dalmanoglu, O. Doğru, On statistical approximation properties of Kantorovich type  $q$ -Bernstein operators, Math. Comput. Model 52(2010) 760-771.
- [19] A Ersan, O. Doğru, statistical approximation properties of  $q$ -Bleimann, Butzer and Hahn operators, Math. Comput. Modell. 49 (2009) 1595-1606.
- [20] C.V. Muraru, Note on  $q$ -Bernstein-Schurer operators, Studia Univ. Babeş-Bolyai, Mathematica 56(2)(2011)489-495.
- [21] M. Örkücü, O. Doğru, Statistical approximation of a kind of Kantorovich type  $q$ -Szász-Mirakjan operators, Nonlinear Anal. 75 (5)(2012) 2874-2882.